Some results on dynamical black holes

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Abstract

We give indications that outer future trapping horizons play a role in the particular semi-classical instability of an evolving black hole that produces the Hawking's radiation. These are obtained with the use of the Hamilton-Jacobi tunneling method. It automatically selects one special expression for the surface gravity of a changing horizon, the one defined a decade ago by Hayward using Kodama's theory of spherically symmetric gravitational fields. The method also applies to point masses embedded in an expanding universe and to general, spherically symmetric black holes. The local surface gravity solves a puzzle concerning the charged stringy black holes, namely that it vanishes in the extremal limit, whereas the Killing global gravity does not.

Devoted to Prof. I. Brevik on the occasion of his 70th birthday.

1 Introduction

It has long been felt that the usual semi-classical treatment of stationary black holes (abbrev. BHs) should be extended to cover at least slowly changing, or evolving black holes. By this expression we mean black holes that can still be described in terms of few multiplole moments such as mass, angular momentum and the charges associated to local gauge symmetries, except that the parameters and the causal structure change with time either because matter and gravitational radiation fall in, or because there operate a Hawking's process of quantum evaporation or finally because the hole is actually immersed in a slowly expanding universe. A technical definition of a "slowly varying BH" can be given in some cases, an example being the Booth-Fairhurst slowly evolving horizon, but in general it depends on the actual physical processes involved. For example, in the case of Hawking's evaporation, conditions for slowness in the presence of a near-horizon viscous fluid have been given by Brevik [1] in an interesting attempt to generalize 't Hooft's model of the self-screening Hawking atmosphere (quantum corrections to this model can be found in [2]). In general it is understood that the black hole temperature is to be be much smaller than the Planck mass, or equivalently the mass $M \gg M_P = G^{-1/2} \sim 10^{19}$ Gev, while in order to study the effects of the expansion the Hubble rate H^{-1} should dominate over the black hole emission/absorption rate.

One surprising aspect of the semi-classical results obtained so far, is that the radiation caused by the changing metric of the collapsing star approaches a steady outgoing flux for large times, implying a drastic violation of energy conservation if one neglects the back reaction of the quantum radiation on the structure of spacetime. But the back reaction problem has not been

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solved yet in a satisfactory way. As pointed out by Fredenhagen and Haag long ago [3], if the back reaction is taken into account by letting the mass of the black hole to change with time, then the radiation will possibly originate from the *surface* of the black hole at all times after its formation.

This poses the question: *what is and where is the surface of a dynamical black hole*? This issue always baffled scientists from the very beginning and produced several reactions during the nineties, which eventually culminated with the notion of outer trapping horizons by Hayward[4] and the isolated [5] and dynamical horizons of Ashtekar and Krishnan [6, 7] (a fine review is in [8]). Thus one is concerned to show, in the first place, what kind of a surface a dynamical horizon can be and also which definition can capture a useful local notion of such a surface, and then what sort of instability, if any, really occurs near the horizon of the changing black hole. This question was non trivial since a changing horizon is typically embedded in a dynamical space-time and it is not even expected to be a null hypersurface, although it is still one of infinite red shift.

We analyze this question for a class of dynamical black hole solutions that was inspired by problems not directly related to black hole physics, although these were subsequently reconsidered in the light of the black hole back reaction problem in the early Eighties. The metrics we shall consider are the Vaidya radiating metric [9], as revisited by J. Bardeen [10] and J. York [11], together with what really is a fake dynamical black holes, the McVittie solution representing, in author's mind, a point mass in cosmology [12]. We shall indicate how the results can be extended to all dynamical, spherically symmetric solutions admitting a possibly dynamical future outer trapping horizon.

2 Horizons

After the time lasting textbook definition of the event horizon (abbr. EH) to be found in the Hawking & Ellis renowned book [13], several quasi-local notions of dynamical horizons appeared in the literature (a nice review is in [14]) , perhaps starting with the perfect horizons[1] of Hájiček [15] and the apparent horizons (AHs, boundaries of trapped 3-dimensional space-like regions) of Hawking-Ellis themselves. But the former only applied to equilibrium BHs and the existence of the latter is tie to a partial Cauchy surface so it represents only a "localization in time". Moreover it has proven not possible to formulate thermodynamical laws for AHs akin to those holding for event horizons.

The first succesfull attempt to go beyond the limitations imposed either by the istantaneous character of the apparent horizons or by the global, teleological nature of the event horizons is due to S. Hayward. His concept of a future outer trapping horizon (FOTH) then evolved either into some less constrained definition, like the Ashtekar-Krishnan dynamical horizons (DH), or some specialization like the Booth-Fairhurst slowly evolving FOTH [16]; so an updated (but perhaps partial) list of locally or quasi-locally defined horizons would contain:

(a) Trapping horizons (Hayward [4])

(b) Dynamical horizons (Ashtekar & Krishnan [7, 6])

(c) Non expanding and perfect horizons (Hájiček [15])

(d) Isolated and weakly isolated horizons (Ashtekar et al. [5])

(e) Slowly evolving horizons (Booth & Fairhurst [16])

[1] These are null hypersurfaces whose rays have zero expansion and intersect space-like hypersurfaces in compact sets. All stationary horizons are perfect, but the converse is not true.

In contrast to the old fashioned apparent horizons, these newly defined horizons do not require a space-like hypersurface, no notion of interior and exterior and no conditions referring to infinity (all are non local conditions). Moreover they are not teleological and, given a solution of Einstein equations, one can find whether they exist by purely local computations. Finally, unlike EHs they are related to regions endowed by strong gravitational fields and absent in weak field regions. All quasi-local horizons rely on the local concept of trapped (marginally trapped) surface: this is a space-like closed 2-manifold $\boldsymbol{S}$ such that $\theta_{(\ell)}\theta_{(n)} > 0$, where ℓ, n are the future-directed null normals to $\boldsymbol{S}$, normalized to $\ell \cdot n = -1$, and $\theta_{(\ell)}$, $\theta_{(n)}$ are the respective expansion scalars. We write the induced metric on each $\boldsymbol{S}$ in the form

$$q_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a \tag{2.1}$$

and put $q^{ab} = g^{ab} + \ell^a n^b + \ell^b n^a$, not an inverse. Then q^a_b is the projection tensor to $T_*(\boldsymbol{S})$, the tangent space to $\boldsymbol{S}$. To cover BHs rather than white holes it is further assumed that both expansions are negative (non positive).
The most important quantities associated with the null vector fields ℓ and n are the projected tensor fields $\Theta_{ab} = q^a_n q^b_m \nabla_a l_b$ and $\Phi_{ab} = q^a_n q^b_m \nabla_a n_b$ and their decomposition into symmetric, anti-symmetric and trace part. Their twists are zero since they are normal to $\boldsymbol{S}$. Finally, the expansions are given by

$$\theta_{(\ell)} = q^{ab}\nabla_a \ell_b, \qquad \theta_{(n)} = q^{ab}\nabla_a n_b \tag{2.2}$$

Let us describe the listed horizons in turn, adding comments where it seems appropriate. A black triangle down ▼ will close the definitions.
Future Outer Trapping Horizon: A future outer trapping horizon (FOTH) is a smooth three-dimensional sub-manifold H of space-time which is foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, $t \in \mathbb{R}$, with future-directed null normals ℓ and n such that (i) the expansion $\theta_{(\ell)}$ of the null normal ℓ vanishes, (ii) the expansion $\theta_{(n)}$ of n is negative and (iii) $\mathcal{L}_n \theta_{(\ell)} < 0$. ▼
Condition (i) requires strong fields since certainly $\theta_{(\ell)} > 0$ in weak fields. Condition (ii) is related to the idea that H is of the future type (i. e. a BH rather than a WH); (iii) says that H is of the outer type, since a motion of $\boldsymbol{S}_t$ along n^a makes it trapped. It also distinguishes BH horizons from cosmological ones.
One can always found a scalar field C on H so that

$$V^a = \ell^a - C n^a \quad \text{and} \quad N_a = \ell_a + C n_a\,, \tag{2.3}$$

are respectively tangent and normal to the horizon. Note that $V \cdot V = -N \cdot N = 2C$. Hayward [4] has shown that if the null energy condition holds, then $C \geq 0$ on a FOTH. Thus, the horizon must be either space-like or null, being null iff the shear $\sigma^{(\ell)}_{ab}$ as well as $T_{ab}\ell^a \ell^b$ both vanish across H. Intuitively, H is space-like in the dynamical regime where gravitational radiation and matter are pouring into it and is null when it reaches equilibrium.
The second law of trapping horizon mechanics follows quite easily. If $\sqrt{q}$ is the area element corresponding to the metric q_{ab} on the cross-sections, then

$$\mathcal{L}_V \sqrt{q} = -C\theta_{(n)}\sqrt{q}\,. \tag{2.4}$$

By definition $\theta_{(n)}$ is negative and we have just seen that C is non-negative, so we obtain the local second law: If the null energy condition holds, then the area element $\sqrt{q}$ of a FOTH is non-decreasing along the horizon. Integrating over $\boldsymbol{S}_t$, the same law applies to the total area of the horizon sections. It is non-decreasing and remains constant if and only if the horizon is null.
Dynamical Horizon: a smooth three-dimensional, *space-like* sub-manifold H of space-time is a dynamical horizon (DH) if it can be foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, with

future-directed null normals ℓ and n such that (i) on each leaf the expansion $\theta_{(\ell)}$ of one null normal ℓ^a vanishes, (ii) the expansion $\theta_{(n)}$ of the other null normal n is negative.▼

Like FOTHs, a DH is a space-time notion defined quasi-locally, it is not relative to a space-like hypersurface, it does not refer to ∞, it is not teleological. A space-like FOTH is a DH on which $\mathcal{L}_n\theta_{(\ell)} < 0$; a DH which is also a FOTH will be called a space-like future outer horizon (SFOTH). The DH cannot describe equilibrium black holes since it is space-like by definition, but is better suited to describe how a BH grow in general relativity. Suitable analogues of the laws of black hole mechanics hold for both FOTHs and DHs. Our main interest in the following will be precisely for these local horizons, but for the time being we continue our description.

Perfect and Non-Expanding Horizons: a perfect horizon is a smooth three-dimensional *null* sub-manifold H of space-time with null normal ℓ^a such that $\theta_{(\ell)} = 0$ on H and which intersect space-like hypersurfaces in compact sets. ▼

If in the last clause H is topologically $\mathbb{R} \times S^2$ and moreover the stress tensor T_{ab} is such that $-T^a_b\ell^b$ is future causal for any future directed null normal ℓ^a, then H is called a non-expanding horizon.▼

If X, Y are tangent to a non-expanding horizon we can decompose the covariant derivative

$$\nabla_X Y = D_X Y + N(X,Y)\ell + L(X,Y)n$$

where D_X is the projection of the vector $\nabla_X Y$ onto the spheres $\boldsymbol{S}_t$ in H. If X is tangent to the spheres then D_X is the covariant derivative of the induced metric q_{ab}, and if X is tangent to H one may regard the operator $\widehat{\nabla}_X = D_X + N(X,\cdot)\ell$, acting on vector fields, as a connection on H. If this connection is "time independent" then the geometry of H is time independent too and we have Ashtekar et al. notion of an horizon in isolation.

Isolated Horizon: a non-expanding horizon with null normal ℓ^a such that $[\mathcal{L}_\ell, \widehat{\nabla}_X] = 0$ along H.▼

These horizons are intended to model BHs that are themselves in equilibrium but possibly in a dynamical space-time. For a detailed description of their mathematical properties we refer the reader to Ashtekar-Krishnan's review [8].

The new horizons just introduced all have their own dynamics governed by Einstein eq.s. There exist for them existence and uniqueness theorems [17], formulation of first and second laws [8, 4, 18] and even a "membrane paradigm" analogy. In particular, they carry a momentum density which obey a Navier-Stokes like equation generalizing the classical Damour's equations of EHs, except that the bulk viscosity $\zeta_{FOTH} = 1/16\pi > 0$ [19, 20]. We think Iver would be amused by that.

The new horizons are also all space-like or null, hence it remains to see what is the role they play in the problem of black hole quantum evaporation. In this connection the following notion can be useful.

Time-like Dynamical Horizon: a smooth three-dimensional, *time-like* sub-manifold H of space-time is a time-like dynamical horizon (TDH) if it can be foliated by closed space-like two-manifolds $\boldsymbol{S}_t$, with future-directed null normals ℓ and n such that (i) on each leaf the expansion $\theta_{(\ell)}$ of one null normal ℓ^a vanishes, (ii) the expansion $\theta_{(n)}$ of the other null normal n is strictly negative.▼

Surface Gravities

The surface gravity associated to an event horizon is a well known concept in black hole physics whose importance can be hardly overestimated. Surprisingly, a number of inequivalent definitions beyond the historical one appeared recently (over the last 15 years or so) in the field with various underlying motivations. We have collected the following (we rely on the nice review of Nielsen and Yoon [28]):

1. The historical Killing surface gravity (Bardeen et al. [21], textbooks)
2. Hayward's first definition [4]
3. Mukohyama-Hayward's definition [22]
4. Booth-Fairhurst surface gravity for the evolving horizons [16]
5. The effective surface gravity appearing in Ashtekar-Krishnan [8]
6. The Fodor et al. definition for dynamical spherically symmetric space-times [23]
7. The Visser [24] and Nielsen-Visser [25] surface gravity
8. Hayward's definition [26] using Kodama's theory [27].

We will not spend much time on the various definitions and their motivations except for the last item, which is what the tunneling approach leads to, among other things.

1. The Killing surface gravity is related to the fact that the integral curves of a Killing vector are not affinely parametrized geodesics on the Killing horizon H. Hence

$$K^a \nabla_b K_a \cong \kappa K_a$$

defines the Killing surface gravity κ on H, where $\cong$ means evaluation on the horizon. The Killing field is supposed to be normalized at infinity by $K^2 = -1$. The definition can be extended to EHs that are not Killing horizons, by replacing K with the null generator ℓ of the horizon. However there is no preferred normalization in this case, and this is one reason of the debating question regarding the value of the surface gravity in dynamical situations.

2. Hayward's first definition was motivated by the desire to get a proof of the first law for THs. It is defined without appeal to inaffinity of null geodesics as

$$\kappa \cong \frac{1}{2}\sqrt{-n^a \nabla_a \theta_{(\ell)}} \tag{2.5}$$

and is independent on the parametrization of ℓ^a integral curves, since the evaluation is on a marginal outer surface where $n \cdot \ell = -1$ and $\theta_{(\ell)} = 0$.

3. We leave apart the Mukohyama-Hayward and the Booth-Fairhurst definitions (see **4.**) as they are somewhat more technical and complicated than it is necessary, so we refer the reader to the original papers.

5. Given a weakly isolated horizon H, Ashtekar and Krishnan showed that for any vector field t^a along H with respect to which energy fluxes across H are defined, there is an area balance law that takes the form

$$\delta E^t = \frac{\bar{\kappa}}{8\pi G}\delta A_S + \text{work terms}$$

with an effective surface gravity given by

$$\bar{\kappa} = \frac{1}{2R}\frac{dr}{dR}$$

R is the areal radius of the marginally trapped surfaces, i.e. $A_S = 4\pi R^2$, the function r is related to a choice of a lapse function and finally E^t is the energy associated with the evolution vector field t^a. For a spherically symmetric DH a natural choice would be $r = R$ so $\bar{\kappa} = 1/2R$, just the result for a Schwarschild BH. To illustrate the naturalness of this definition, consider a slowly

changing spherically symmetric BH with mass $M(v)$, where v is a time coordinate. Defining the horizon radius at each time by $R = 2M(v)$ and $A_S = 4\pi R^2$, we can differentiate M to obtain

$$\dot{M} = \frac{\dot{R}}{2} = \frac{1}{2R}\frac{\dot{A}_S}{8\pi} \implies \delta M = \frac{\bar{\kappa}}{8\pi}\delta A_S$$

which is the usual area balance law with surface gravity $\bar{\kappa} = 1/2R = 1/4M$. Consider, however, the more general possibility where the horizon is at $R = 2M(v, R)$, as it happens for example in the Vaidya-Bardeen metric. The same computation leads to

$$\dot{M} = \frac{1}{2R}\left(1 - 2M'\right)\frac{\dot{A}_S}{8\pi} \implies \kappa \cong \frac{1}{4M}\left(1 - 2M'\right) \tag{2.6}$$

a prime denoting the radial derivative. Thus naturalness is not a decisive criterion in this case.
6. The Fodor et al. definition looks like the Killing form of surface gravity in that $\kappa\ell^b = \ell^a\nabla_a\ell^b$, where now ℓ^a is an outgoing null vector orthogonal to a trapped or marginally trapped surface. This is because, as a rule, such null vectors are not affinely parametrized, although they can always be so parametrized that $\kappa = 0$. So one needs to fix the parametrization: Fodor et al. choose

$$\kappa = -n^a\ell^b\nabla_b\ell_a$$

with n^a affinely parametrized and normalized to $n \cdot t = -1$ at space-like infinity, t^a being the asymptotic Killing field. Note that this definition is non local but looks like as a natural generalization of the Killing surface gravity.
7. We postpone the discussion of the Visser and Visser-Nielsen surface gravity to the next section.
8. Finally we have a local geometrical definition of this quantity for the trapping horizon of a spherically symmetric black hole as [26] follows. One can introduce local null coordinates $x^\pm$ in a tubular neighborhood of a FOTH such that $n = -g^{+-}\partial_-$ and $\ell = \partial_+$: then (shortening $\theta_{(\ell)} = \theta_+$, $\theta_{(n)} = \theta_-$)

$$\kappa = \frac{1}{2}\left(g^{+-}\partial_-\theta_+\right)_{|\theta_+=0} \tag{2.7}$$

Later we will show that this κ fixes the expansion of the metric near the trapping outer horizon along a future null direction. The definition may look somewhat artificial, but in fact it is very natural and connected directly with what is known for the stationary black holes. To see this one notes, following Kodama [27], that any spherically symmetric metric admits a unique (up to normalization) vector field K^a such that K^aG_{ab} is divergence free, where G_{ab} is the Einstein tensor; for instance, using the double-null form, one finds $K = -g^{+-}(\partial_+r\partial_- - \partial_-r\partial_+)$. The defining property of K shows that it is a natural generalization of the time translation Killing field of a static black hole. Moreover, by Einstein equations K_aT^{ab} will be conserved so for such metrics there exists a natural localizable energy flux and its conservation law. Now consider the expression $K^a\nabla_{[b}K_{a]}$: it is not hard to see that on H it is proportional to K_b. The proportionality factor, a function in fact, is the surface gravity: $K^a\nabla_{[b}K_{a]} = -\kappa K_b$. For a Killing vector field ∇_bK_a is anti-symmetric so the definition reduces to the usual one.

3 Two examples: Vaidya and McVittie's metrics

We consider first spherically symmetric spacetimes which outside the horizon (if there is one) are described by a metric of the form

$$ds^2 = -e^{2\Psi(r,v)}A(r,v)dv^2 + 2e^{\Psi(r,v)}dvdr + r^2dS^2\,. \tag{3.1}$$

where the coordinate r is the areal radius commonly used in relation to spherical symmetry and v is intended to be an advanced null coordinate. In an asymptotically flat context one can always write (we use geometrized units in which the Newton constant $G = 1$)

$$A(r,v) = 1 - 2m(r,v)/r \tag{3.2}$$

which defines the active mass. This metric was first proposed by Vaidya [9], and studied in an interesting paper during the classical era of black hole physics by Lindquist et al [29]. It has been generalized to Einstein-Maxwell systems and de Sitter space by Bonnor-Vaidya and Mallet, respectively [30]. It was then extensively used by Bardeen [10] and York [11] in their semi-classical analysis of the back reaction problem. We will call it the Vaidya-Bardeen metric. A cosmological constant can be introduced by setting

$$A(r,v) = 1 - 2m(r,v)/r - r^2/L^2 \tag{3.3}$$

where $L^{-2} \propto \Lambda$. If one wishes the metric can also be written in double-null form. In the (v,r)-plane one can introduce null coordinates $x^{\pm}$ such that the dynamical Vaidya-Bardeen space-time may be written as

$$ds^2 = -2f(x^+,x^-)dx^+dx^- + r^2(x^+,x^-)dS^2_{D-2}\,, \tag{3.4}$$

for some differentiable function f. The remaining angular coordinates contained in dS^2 do not play any essential role. In the following we shall use both forms of the metric, depending on computational convenience. The field equations of the Vaidya-Bardeen metric are of interest. They read

$$\frac{\partial m}{\partial v} = 4\pi r^2 T^r_{\ v}, \quad \frac{\partial m}{\partial r} = -4\pi r^2 T^v_{\ v}, \quad \frac{\partial \Psi}{\partial r} = 4\pi r e^{\Psi} T^v_{\ r} \tag{3.5}$$

The stress tensor can be written as

$$T_{ab} = \frac{\dot{m}}{4\pi r^2}\nabla_a v \nabla_b v - \frac{m^{'}}{2\pi r^2}\nabla_{(a} r \nabla_{b)} v \tag{3.6}$$

If m only depends on v it describes a null fluid and obeys the dominant energy condition if $\dot{m} > 0$.

The second example we are interested in is the McVittie solution [12] for a point mass in a Friedmann-Robertson-Walker flat cosmology. In D-dimensional spacetime in isotropic spatial coordinates it is given by [31]

$$ds^2 = -A(\rho,t)dt^2 + B(\rho,t)\left(d\rho^2 + \rho^2 dS^2_{D-2}\right) \tag{3.7}$$

with

$$A(\rho,t) = \left[\frac{1 - \left(\frac{m}{a(t)\rho}\right)^{D-3}}{1 + \left(\frac{m}{a(t)\rho}\right)^{D-3}}\right]^2, \qquad B(\rho,t) = a(t)^2\left[1 - \left(\frac{m}{a(t)\rho}\right)^{D-3}\right]^{2/(D-3)}. \tag{3.8}$$

When the mass parameter $m = 0$, it reduces to a spatially flat FRW solution with scale factor $a(t)$; when $a(t) = 1$ it reduces to the Schwarzschild metric with mass m. In four dimensions this solution has had a strong impact on the general problem of matching the Schwarzschild solution with cosmology, a problem faced also by Einstein and Dirac. Besides McVittie, it has been extensively studied by Nolan in a series of papers [32]. To put the metric in the general form of Kodama theory, we use what may be called the Nolan gauge, in which the metric reads

$$ds^2 = -\left(A_s - H^2(t)r^2\right)dt^2 + A_s^{-1}dr^2 - 2A_s^{-1/2}H(t)r\,dr dt + r^2 dS^2_{D-2} \tag{3.9}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter and, for example, in the charged 4-dimensional case,

$$A_s = 1 - 2m/r + q^2/r^2 \tag{3.10}$$

or in D dimension $A_s = 1 - 2m/r^{D-3} + q^2/r^{2D-6}$. In passing to the Nolan gauge a choice of sign in the cross term $drdt$ has been done, corresponding to an expanding universe; the transformation $H(t) \to -H(t)$ changes this into a contracting one. In the following we shall consider $D = 4$ and $q = 0$; then the Einstein-Friedmann equations read

$$3H^2 = 8\pi\rho\,, \qquad 2A_s^{-1/2}\dot{H}(t) + 3H^2 = -8\pi p\,. \tag{3.11}$$

It follows that $A_s = 0$, or $r = 2m$, is a curvature singularity. In fact, it plays the role that $r = 0$ has in FRW models, namely it is a big bang singularity. When $H = 0$ one has the Schwarzschild solution. Note how the term H^2r^2 in the metric strongly resembles a varying cosmological constant; in fact for H a constant, it reduces to the Schwarzschild-de Sitter solution in Painlevé coordinates. As we will see, the McVittie solution possesses in general both apparent and trapping horizons, and the spacetime is dynamical. However, it is really not a dynamical black hole in the sense we used it above, since the mass parameter is strictly constant: for this reason we called it a fake dynamical BH. This observation prompts one immediately for an obvious extension of the solution: to replace the mass parameter by a function of time and radius, but this will not be pursued here.

The study of black holes requires also a notion of energy; the natural choice would be to use the charge associated to Kodama conservation law, but this turns out to be the Misner-Sharp energy, which for a sphere with areal radius r is the same as the Hawking mass [33], given by $E = r(1 - 2^{-1}r^2g^{+-}\theta_+\theta_-)/2$. Using the metric (3.1) an equivalent expression is

$$g^{\mu\nu}\partial_\mu r\partial_\nu r = 1 - 2E/r \tag{3.12}$$

In this form it is clearly a generalization of the Schwarzschild mass. As we said, E is just the charge associated to Kodama conservation law; as showed by Hayward [34], in vacuum E is also the Schwarzschild energy, at null infinity it is the Bondi-Sachs energy and at spatial infinity it reduces to the ADM mass.

Let us apply this general theory to the two classes of dynamical BH we have considered. Using Eq. (3.1), we have $\theta_{(\ell)} = A(r,v)/2r$. The condition $\theta_{(\ell)} = 0$ leads to $A(r_h, v) = 0$, which defines a curve $r_h = r_h(v)$ giving the location of the horizon; it is easy to show that $\theta_- < 0$, hence the horizon is of the future type. Writing the solution in the Vaidya-Bardeen form, that is with $A(r,v) = 1 - 2m(r,v)/r$, the Misner-Sharp energy of the black hole is $E = m(r_h(v), v)$, and the horizon will be outer trapping if $m^{'}(r_h, v) < 1/2$, a prime denoting the radial derivative. The geometrical surface gravity associated with the Vaidya-Bardeen dynamical horizon is

$$\kappa(v) \cong \frac{A^{'}(r,v)}{2} = \frac{m(r_h,v)}{r_h^2} - \frac{m^{'}(r_h,v)}{r_h} = \frac{1}{4m}\left(1 - 2m^{'}\right) \tag{3.13}$$

the same as Eq. (2.6), where $m \equiv m(r_h, v)$. We see the meaning of the outer trapping condition: it ensures the positivity of the surface gravity.

As a comparison, Hayward's first definition would give $\kappa = \sqrt{1 - 2m^{'}}/4m$, while Fodor et al. expression is

$$\kappa = \frac{2^{\Psi}}{4m}(1 - 2m^{'}) + \dot{\Psi} \tag{3.14}$$

The effective surface gravity of Ashtekar-Krishnan simply is $\kappa = 1/4m$, everything being evaluated on the horizon. Note that some of them are not correct for the Reissner-Nordström black

hole. We also note that κ (3.13) is inequivalent to the Nielsen-Visser surface gravity, which in these coordinates takes the form

$$\tilde{\kappa} = \frac{1}{4m}\left(1 - 2m^{'} - e^{-\Psi}\dot{m}\right) \tag{3.15}$$

though they coincide in the static case. Also, both are inequivalent to the Visser surface gravity $e^{\Psi}\tilde{\kappa}$, which was derived as a temperature by essentially the same tunneling method as discussed below, but in Painlevé-Gullstrand coordinates. Part of the difference can be traced to a different choice of time.
In the case of McVittie BHs, we obtain

$$\theta_{\pm} = \pm(\sqrt{A_s} \mp Hr)/2rf_{\pm}$$

where the functions $f_{\pm}$ are integrating factors determining null coordinates $x^{\pm}$ such that $dx^{\pm} = f_{\pm}\left[\left(\sqrt{A_s} \pm Hr\right)dt \pm A_s^{-1/2}dr\right]$. One may compute from this the dual derivative fields $\partial_{\pm}$. The future dynamical horizon defined by $\theta_{+} = 0$, has a radius which is a root of the equation $\sqrt{A_s} = Hr_h$, which in turn implies $A_s = H^2r_h^2$. Hence the horizon radius is a function of time. The Misner-Sharp mass and the related surface gravity are

$$E = m + \frac{1}{2}H(t)^2r_h^3 \tag{3.16}$$

$$\kappa(t) = \frac{m}{r_h^2} - H^2r_h - \frac{\dot{H}}{2H} = \frac{E}{r_h^2} - \frac{3}{2}H^2r_h - \frac{\dot{H}}{2H} \tag{3.17}$$

Note that $E = r_h/2$. In the static cases everything agrees with the standard results. The surface gravity has an interesting expression in terms of the sources of Einstein equations and the Misner-Sharp mass. Let $\tilde{T}$ be the reduced trace of the stress tensor in the space normal to the sphere of symmetry, evaluated on the horizon H. For the Vaidya-Bardeen metric it is, by Einstein's equations (3.5),

$$\tilde{T} = T_v^v + T_r^r = -\frac{1}{2\pi r_h}\frac{\partial m}{\partial r}_{|r=r_h}$$

For the McVittie's solution, this time by Friedmann's equations (3.11) one has

$$\tilde{T} = -\rho + p = -\frac{1}{4\pi}\left(3H^2 + \frac{\dot{H}}{Hr_h}\right)$$

We have then the mass formula

$$\frac{\kappa A_H}{4\pi} = E + 2\pi r_h^3\tilde{T} \tag{3.18}$$

where $A_H = 4\pi r_h^2$. It is worth mentioning the pure FRW case, i.e. $A_s = 1$, for which $\kappa(t) = -\left(H(t) + \dot{H}/2H\right)$. One can easily see that (3.18) is fully equivalent to Friedmann's equation. We feel that these expressions for the surface gravity are non trivial and display deep connections with the emission process. Indeed it is the non vanishing of κ that is connected with the imaginary part of the action of a massless particle, as we are going to show in the next section.

4 Tunneling and instability

The essential property of the tunneling method is that the action I of an outgoing massless particle emitted from the horizon has an imaginary part which for stationary black holes is $\Im I = \pi\kappa^{-1}E$, where E is the Killing energy and κ the horizon surface gravity. The imaginary part is obtained by means of Feynman $i\epsilon$-prescription, as explained in [35, 36]. As a result the particle production rate reads $\Gamma = \exp(-2\Im I) = \exp(-2\pi\kappa^{-1}E)$. One then recognizes the Boltzmann factor, from which one deduces the well-known temperature $T_H = \kappa/2\pi$. Moreover, an explicit expression for κ is actually obtained in terms of radial derivatives of the metric on the horizon.

Let us consider now the case of a dynamical black hole in the double-null form [37]. We have for a massless particle along a radial geodesic the Hamilton-Jacobi equation $\partial_+ I \partial_- I = 0$. Since the particle is outgoing $\partial_- I$ is not vanishing, and we arrive at the simpler condition $\partial_+ I = 0$. First, let us apply this condition to the Vaidya-Bardeen BH. One has then

$$2e^{-\Psi(r,v)}\partial_v I + A(r,v)\,\partial_r I = 0\,. \tag{4.1}$$

Since the particle will move along a future null geodesic, to pick the imaginary part we expand the metric along a future null direction starting from an arbitrary event $(r_h(v_0), v_0)$ on the horizon, i.e. $A(r_h(v_0), v_0) = 0$. Thus, shortening $r_h(v_0) = r_0$, we have $A(r,v) = \partial_r A(r_0, v_0)\Delta r + \partial_v A(r_0, v_0)\Delta v + \cdots = 2\kappa(v_0)(r - r_0) + \ldots$, since along a null direction at the horizon $\Delta v = 0$, according to the metric (3.1); here $\kappa(v_0)$ is the surface gravity, Eq. (3.13). From (4.1) and the expansion, $\partial_r I$ has a simple pole at the event (r_0, v_0); as a consequence

$$\Im I = \Im \int \partial_r I dr = -\Im \int dr \frac{2e^{-\Psi(r,v)}\partial_v I}{A'(r_0, v_0)(r - r_0 - i0)} = \frac{\pi\omega(v_0)}{\kappa(v_0)}\,. \tag{4.2}$$

where $\omega(v_0) = e^{-\Psi(r_0,v_0)}\partial_v I$, is to be identified with the energy of the particle at the time v_0. Note that the Vaidya-Bardeen metric has a sort of gauge invariance due to conformal reparametrizations of the null coordinate v: the map $v \to \tilde{v}(v)$, $\Psi(v,r) \to \tilde{\Psi}(\tilde{v}, r) + \ln(\partial\tilde{v}/\partial v)$ leaves the metric invariant, and the energy is gauge invariant too. Thus we see that the Hayward-Kodama surface gravity appears to be relevant to the process of particles emission. The emission probability, $\Gamma = \exp(-2\pi\omega(v)/\kappa(v))$, has the form of a Boltzmann factor, suggesting a locally thermal spectrum.

For the McVittie's BH, the situation is similar. In fact, the condition $\partial_+ I = 0$ becomes

$$\partial_r I = -F(r,t)^{-1}\partial_t I$$

where

$$F(r,t) = \sqrt{A_s(r)}\left(\sqrt{A_s(r)} - rH(t)\right)$$

As before, we pick the imaginary part by expanding this function at the horizon along a future null direction, using the fact that for two neighbouring events on a null direction in the metric (3.9), one has $t - t_0 = (2H_0^2 r_0^2)^{-1}(r - r_0)$, where $H_0 = H(t_0)$. We find the result

$$F(r,t) = \left(\frac{1}{2}A_s'(r_0) - r_0 H_0^2 - \frac{\dot{H}_0}{2H_0}\right)(r - r_0) + \cdots = \kappa(t_0)(r - r_0)\ldots \tag{4.3}$$

where this time $r_0 = r_h(t_0)$. From this equation we see that $\partial_r I$ has a simple pole at the horizon; hence, making use again of Feynman $i\epsilon$-prescription, one finds $\Im I = \pi\kappa(t_0)^{-1}\omega(t_0)$, where $\omega(t) = \partial_t I$ is again the energy at time t, in complete agreement with the geometric evaluation of the previous section. Obviously, if κ vanishes on the horizon there is no simple

pole and the black hole should be stable[2]. The kind of instability producing the Hawking flux for stationary black holes evidently persists in the dynamical arena, and so long as the evolution is sufficiently slow the black hole seems to evaporate thermally. Note that the imaginary part, that is the instability, is attached to the horizon all the time, confirming the Fredenhagen-Haag suggestion quoted in the introduction. It is worth mentioning the role of κ in the analogue of the first law for dynamical black holes (contributions to this problem for Vaidya black holes were given in [39]). Using the formulas of the projected stress tensor $\tilde{T}$ given above, and the expression of the Misner-Sharp energy, one obtains the differential law

$$dE = \frac{\kappa\, dA_H}{8\pi} - \frac{\tilde{T}}{2}\, dV_H \tag{4.4}$$

provided all quantities were computed on the trapping horizon. Here $A_H = 4\pi r_H^2$ is the horizon area and $V_H = 4\pi r_H^3/3$ is a formal horizon volume. If one interprets the "d" operator as a derivative along the future null direction one gets Hayward's form of the first law. But one can also interpret the differential operation more abstractly, as referring to an ensemble. Indeed, to obtain Eq. (4.4) it is not necessary to specify the meaning of the "d". It is to be noted that the same law can be proved with other, inequivalent definitions of the surface gravity, even maintaining the same meaning for the energy. Thus other considerations are needed to identify one: the tunneling method has made one choice.

As thoroughly discussed in Hayward et al. [40], Eq. (3.1) is actually the most general form of a spherically symmetric metric, so the above calculations works throughout. Of course $\kappa > 0$ if the trapping horizon is of the outer type. Thus the method has derived a positive temperature if and only if there is a future outer trapping horizon.

Extremal limit

We discuss only an example, the charged stringy black hole, which represents a non-vacuum solution of Einstein-Maxwell dilaton gravity in the string frame [41, 42]:

$$ds^2 = r^2 d\Omega^2 + \frac{dr^2}{(1 - a/r)\,(1 - b/r)} - \left(\frac{1 - a/r}{1 - b/r}\right) dt^2 \tag{4.5}$$

where $a > b > 0$. The horizon radius is $r = a$.

For this example, the extremal limit as defined by global structure is $b \to a$. The Killing surface gravity $\kappa_\infty \cong 1/2a$ does not vanish in this limit. Garfinkle et al. [42] noted this as puzzling, since extremal black holes are expected to be zero-temperature objects.

Remarkably, the geometrical surface gravity (3.13)

$$\kappa \cong \frac{a - b}{2a^2} \tag{4.6}$$

vanishes in the extremal limit. Thus the gravitational dressing effect lowers the temperature to its theoretically expected value.

We conjecture that this is true in general. Indeed, past experience with extremal black holes showed that the horizon of these objects is not only a zero but also a minimum of the expansion $\theta_+ = \partial_+ A/A$ of the radially outgoing null geodesics, θ_+ becoming positive again on crossing the horizon. Thus $\partial_-\theta_+ \cong 0$ should be the appropriate definition of an extremal black hole. Since $\kappa = -e^{-2\varphi}\partial_-\partial_+ r$, this is equivalent to $\kappa = 0$.

[2]However, charged extremal black holes can radiate [38].

References

[1] I. Brevik, Phys. Rev. D **61** (2000) 124017 [arXiv:gr-qc/9906060].

[2] S. Nojiri & S. D. Odintsov, Int. J. Mod. Phys. A **15** (2000) 989 [arXiv:hep-th/9905089].

[3] K. Fredenhagen & R. Haag, Commun. Math. Phys. **127**, 273 (1990).

[4] S. Hayward, Phys. Rev. **49**, 6467 (1994); Class. Quantum Grav. **11**, 3025 (1994).

[5] A. Ashtekar, C. Beetle & S. Fairhurst, Class. Quant. Grav. **16** (1999) L1 [arXiv:gr-qc/9812065].

[6] A. Ashtekar & B. Krishnan, Phys. Rev. D **68** (2003) 104030 [arXiv:gr-qc/0308033].

[7] A. Ashtekar & B. Krishnan, Phys. Rev. Lett. **89**, 261101 (2002; Phys. Rev. **D 68**, 104030 (2003).

[8] A. Ashtekar & B. Krishnan, Living Rev. Rel. **7** (2004) 10 [arXiv:gr-qc/0407042].

[9] P. C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951); P. C. Vaidya, Nature **171**, 260 (1953); V.V. Narlikar & P.C. Vaidya, Nature **159**, 642 (1947).

[10] J.M. Bardeen, Phys. Rev. Letters **46**, 382 (1981).

[11] J. W. York, Phys. Rev. **D 28**, 2929 (1983).

[12] G. C. McVittie , Mon. Not. R. Astronomic Soc. **93**, 325 (1933).

[13] S. W. Hawking & G. F. R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press 1973).

[14] E. Gourgoulhon & J. L. Jaramillo, New Astron. Rev. **51**, 791 (2008) [arXiv:0803.2944 [astro-ph]].

[15] P. Hájiček, Comm. Math. Phys. **34**, 37 (1973); Lett. Nuovo Cimento, **6**, 157 (1973).

[16] I. Booth & S. Fairhurst, Phys. Rev. Lett. **92**, 011102 (2004).

[17] L. Andersson, M. Mars & W. Simon, Phys. Rev. Lett. **95** (2005) 111102 [arXiv:gr-qc/0506013].

[18] S. A. Hayward, Phys. Rev. D **70** (2004) 104027 [arXiv:gr-qc/0408008].

[19] E. Gourgoulhon, Phys. Rev. D **72** (2005) 104007.

[20] E. Gourgoulhon & J. L. Jaramillo, Phys. Rev. D **74** (2006) 087502.

[21] J. M. Bardeen, B. Carter & S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).

[22] S. Mukohyama & S. A. Hayward, Class. Quant. Grav. **17** (2000) 2153 [arXiv:gr-qc/9905085].

[23] G. Fodor, K. Nakamura, Y. Oshiro & A. Tomimatsu, Phys. Rev. D **54** (1996) 3882 [arXiv:gr-qc/9603034].

[24] M. Visser, Int. J. Mod. Phys. D **12** (2003) 649.

[25] A. B. Nielsen & M. Visser, Class. Quant. Grav. **23** (2006) 4637.

[26] S. A. Hayward, Class. Quant. Grav. **15** (1998) 3147 [arXiv:gr-qc/9710089].

[27] H. Kodama, Prog. Theor. Phys. **63**, 1217 (1980).

[28] A. B. Nielsen & J. H. Yoon, Class. Quant. Grav. **25** (2008) 085010 [arXiv:0711.1445 [gr-qc]].

[29] R. W. Lindquist, R. A. Schwartz, C. W. Misner, Phys. Rev. **137**, 1364 (1965).

[30] W. B. Bonnor, P. C. Vaidya, Gen. Rel. Grav. **1**, 127, (1970); R. L. Mallet, Phys. Rev. **D31**, 416 (1985).

[31] C.J. Gao, Class. Quant. Grav. **21**, 4805 (2004).

[32] B. C. Nolan, Phys. Rev. D **58**, 064006 (1998); B. C. Nolan, Class. Quant. Grav. **16**, 1227 (1999); B. C. Nolan, Class. Quant. Grav. **16**, 3183 (1999).

[33] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964); S. W. Hawking, J. Math. Phys. **9**, 598 (1968).

[34] S. A. Hayward, Phys. Rev. D **53**, 1938 (1996).

[35] M. K. Parikh & F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).

[36] M. Angheben, M. Nadalini, L. Vanzo & S. Zerbini, JHEP **0505**, 014 (2005); M. Nadalini, L. Vanzo & S. Zerbini, J. Physics A Math. Gen. **39**, 6601 (2006).

[37] R. Di Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini and G. Zoccatelli, Phys. Lett. B **657** (2007) 107 [arXiv:0707.4425 [hep-th]].

[38] L. Vanzo, Phys. Rev. D **55**, 2192 (1997).

[39] Li Xiang, Zhao Zheng, Phys. Rev. **D 62**, 104001-1 (2000); Xiaoning Wu, Sijie Gao, Phys. Rev. **D75**, 044027 (2007); Ren Ji-Rong, Li Ran, arXiv:0705.4339 [gr-qc].

[40] S. A. Hayward, R. Di Criscienzo, M. Nadalini, L. Vanzo & S. Zerbini, arXiv:0806.0014 [gr-qc]

[41] G. W. Gibbons, Nucl. Phys. **B 207**, 337 (1982); G. W. Gibbons & K. Maeda, Nucl. Phys. **B 298**, 741 (1988).

[42] D. Garfinkle, G. T. Horowitz & A. Strominger, Phys. Rev. D **43**, 3140 (1991) [Erratum-ibid. D **45**, 3888 (1992)].